LOWER BOUNDS FOR NON STANDARD DETERMINISTIC ESTIMATION

Jerome Galy$^{(1)}$, *Eric Chaumette*$^{(2)}$, *Francois Vincent*$^{(2)}$, *Alexandre Renaux*$^{(3)}$ *and Pascal Larzabal*$^{(4)}$

$^{(1)}$ Université de Montpellier 2, Montpellier, France (jerome.galy@univ-montp2.fr)

$^{(2)}$ ISAE-SUPAERO, Université de Toulouse, 31055 Toulouse Cedex 4, France (eric.chaumette@isae.fr, francois.vincent@isae.fr)

$^{(3)}$ Université Paris-Sud/LSS, 3 Rue Joliot-Curie, Gif-sur-Yvette, France (renaux@lss.supelec.fr)

$^{(4)}$ Université Paris-Sud/SATIE, 61 av. du President Wilson, Cachan, France (pascal.larzabal@satie.ens-cachan.fr)

ABSTRACT

In this paper, non standard deterministic parameters estimation is considered, i.e. the situation where the probability density function (p.d.f.) parameterized by unknown deterministic parameters results from the marginalization of a joint p.d.f. depending on additional random variables. Unfortunately, in the general case, this marginalization is mathematically intractable, which prevents from using the known deterministic lower bounds on the mean-squared-error (MSE). However an embedding mechanism allows to transpose all the known lowers bounds into modified lower bounds fitted with non-standard deterministic estimation, encompassing the modified Cramér-Rao / Bhattacharyya bounds and hybrid lower bounds.

Index Terms— Deterministic parameter estimation, estimation error lower bounds

1. INTRODUCTION

As introduced in [1, p53], a model of the general estimation problem has the following four components: 1) a parameter space Θ, 2) an observation space Ω, 3) a probabilistic mapping from vector parameters space Θ to observation space Ω, that is the probability law that governs the effect of a vector parameters value on the observation and, 4) an estimation rule. In many estimation problems, the probabilistic mapping results from a two steps probabilistic mechanism, illustrated by the observation model: $\mathbf{x} = \mathbf{b}(\omega)s + \mathbf{n}$,where $\mathbf{x}$ is the vector of observations of size M, s is a complex amplitude, $\mathbf{b}(\,)$ is a vector of M parametric functions depending on a parameter ω, $\mathbf{n}$ are known complex circular noises independent of s. In a first step, the centred random amplitude s is drawn according to a p.d.f. conditioned on its mean power σ_s^2: $p\left(s|\sigma_s^2\right)$. In a second step, the signal of interest is embedded in noise: $s \rightarrow \mathbf{x} = \mathbf{b}(\omega)s \rightarrow \mathbf{x} = \mathbf{b}(\omega)s + \mathbf{n}$, leading to the probabilistic mapping $\boldsymbol{\theta}^T = \left(\sigma_s^2, \omega\right) \in \Theta \rightarrow \mathbf{x} \in \Omega$ characterized by:

$$p(\mathbf{x}|\boldsymbol{\theta}) = \int p(\mathbf{x}, s|\boldsymbol{\theta})\, ds,\ p(\mathbf{x}, s|\boldsymbol{\theta}) = p(\mathbf{x}|s;\omega)\, p\left(s|\sigma_s^2\right) \quad (1a)$$

As illustrated in (1a), in probability theory, the distribution of the marginal variables $(\mathbf{x})$ is obtained by marginalizing over the distribution of the variables being discarded (s), and the discarded variables are said to have been marginalized out. It is not said that the discarded variables should be regarded as unknown (nuisance) random parameters which estimation could be of interest. Therefore, deterministic estimation problems can be divided into two subsets: the subset of "standard" problems for which a closed-form expression of $p(\mathbf{x}|\boldsymbol{\theta})$ is available, and the subset of "non standard" problems for which only an integral form of $p(\mathbf{x}|\boldsymbol{\theta})$ is available. For a long time, the open literature on lower bounds on the MSE has remained focused on standard estimation [2]-[16]. It is likely that the first attempt to tackle the "non standard" case was addressed by Miller and Chang [17] who introduced a so-called modified Cramér-Rao bound. For their particular problem of interest, it made sense to regard the marginalization (1a) as a probabilistic modelling of the estimation of unknown parameters $(\boldsymbol{\theta})$ from noisy measurements $(\mathbf{x})$ incorporating random nuisance parameters (s). Unfortunately [17] does not address the problem of finding a lower bound on the MSE of a locally-best unbiased estimator as defined by Barankin in its seminal work [7], generalizing the earlier works of Fréchet, Darmois, Cramér, Rao and Battacharayya [2]-[6]. However, the setting introduced in [17] has been replicated and repeated in [18][19] in order to compute a "true" modified CRB (MCRB) for unbiased estimates exploiting (1a). As a consequence, the MCRB introduced in [18][19] is the first[1] lower bound (but not the only one [22]) for unbiased estimates in non-standard estimation and has been proven to be useful in many applications [18]-[29]

As a contribution, we propose a simple approach to derive lower bounds on the MSE for unbiased estimates in non-standard estimation exploiting the general form of the marginalization formula (1a):

$$p(\mathbf{x}|\boldsymbol{\theta}) = \int_{\Pi_{\boldsymbol{\theta}_r|\mathbf{x}}} p(\mathbf{x}, \boldsymbol{\theta}_r|\boldsymbol{\theta})\, d\boldsymbol{\theta}_r,\ \mathbf{x} \in \Omega,\ \boldsymbol{\theta}_r \in \Pi_{\boldsymbol{\theta}_r},\ \boldsymbol{\theta} \in \Theta \quad (1b)$$

without any reference to extraneous or nuisance random parameters. The main result is that the lack of a closed-form for marginal p.d.f. $p(\mathbf{x}|\boldsymbol{\theta})$ compels to embed[2] the initial observation space Ω into $\Omega \times \Pi_{\boldsymbol{\theta}_r}$ and to consider estimation rules from $\Omega \times \Pi_{\boldsymbol{\theta}_r}$, leading to the derivation of lower bounds of unbiased estimator $\widehat{\boldsymbol{\theta}}(\mathbf{x}, \boldsymbol{\theta}_r)$ of $\boldsymbol{\theta}$. This embedding mechanism allows to transpose all the lowers bounds derived in standard estimation (briefly overviewed in section 2) into modified lower bounds fitted with non-standard estimation (see section 3). Interestingly enough, tighter modified lower bounds can be easily obtained (see §3.1 and §3.2) which appear to be the "deterministic part" of hybrid lower bounds (see §3.3), that is the part of the hybrid lower bounds bounding the deterministic parameters, but, without any regularity condition on the (nuisance) random vector estimates. Last, the proposed rationale not only proves straightforwardly the looseness of any modified lower bound (including thus

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[1] Note that CRB's for synchronization parameter estimation that have been derived earlier in [20][21] are in fact MCRBs.

[2] One space $\digamma$ is embedded in another space Υ when the properties of Υ restricted to $\digamma$ are the same as the properties of $\digamma$.

the hybrid lower bounds) but also provides a very general "tightness condition" required to obtain a modified lower bound as tight as the standard lower bound (see §3.4). The results introduced in the following are also of interest if a closed-form of $p(\mathbf{x}|\theta)$ does exist but the resulting expression is intractable to derive lower bounds.
For the sake of simplicity we will focus on the estimation of a single unknown real parameter θ, although the results are easily extended to the estimation of multiple functions of multiple parameters [30].

2. LOWER BOUNDS FOR STANDARD ESTIMATION

Let $L^2(\mathbb{C}^M)$ be the real Euclidean space of square integrable real-valued functions over the domain $\mathbb{C}^M$. In the search for a lower bound on the MSE of unbiased estimators, two fundamental properties have been introduced by Barankin [7]. First, the MSE of a particular estimator $\widehat{\theta^0} \triangleq \widehat{\theta^0}(\mathbf{x}) \in L^2(\mathbb{C}^M)$ of θ^0, where θ^0 is a selected value of the parameter θ, can be formulated as:

$$MSE_{\theta^0}\left[\widehat{\theta^0}\right] = \left\|\widehat{\theta^0}(\mathbf{x}) - \theta^0\right\|^2_{\theta^0},$$
$$\langle g(\mathbf{x}) \mid h(\mathbf{x})\rangle_\theta = E_{\mathbf{x}|\theta}\left[g(\mathbf{x})h(\mathbf{x})\right] = \int_{\mathbb{C}^M} g(\mathbf{x})h(\mathbf{x})p(\mathbf{x}|\theta)\,d\mathbf{x}.$$

Second, an unbiased estimator $\widehat{\theta^0}(\mathbf{x})$ should be uniformly unbiased:

$$\forall\theta\in\Theta: E_{\mathbf{x}|\theta}\left[\widehat{\theta^0}(\mathbf{x})\right] = \int_{\mathbb{C}^M}\widehat{\theta^0}(\mathbf{x})p(\mathbf{x}|\theta)\,d\mathbf{x} = \theta. \tag{2a}$$

If $\Omega(\theta) = \left\{\mathbf{x}\in\mathbb{C}^M \mid p(\mathbf{x}|\theta) > 0\right\} \triangleq \Omega \subset \mathbb{C}^M$, i.e. the support of $p(\mathbf{x}|\theta)$ does not depend on θ, then (2a) can be recasted as:

$$\forall\theta\in\Theta: E_{\mathbf{x}|\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x}) - \theta^0\right)\upsilon_{\theta^0}(\mathbf{x};\theta)\right] = \theta - \theta^0, \tag{2b}$$

where $\upsilon_{\theta^0}(\mathbf{x};\theta) = \frac{p(\mathbf{x}|\theta)}{p(\mathbf{x}|\theta^0)}$ denotes the Likelihood Ratio (LR). As a consequence, the locally-best (at θ^0) unbiased estimator in $L^2(\Omega)$ is the solution of a norm minimization under linear constraints:

$$\begin{array}{l}\min\left\{\left\|\widehat{\theta^0}(\mathbf{x}) - \theta^0\right\|^2_{\theta^0}\right\} \text{ under} \\ \qquad\left\langle\widehat{\theta^0}(\mathbf{x}) - \theta^0 \mid \upsilon_{\theta^0}(\mathbf{x};\theta)\right\rangle_{\theta^0} = \theta - \theta^0, \forall\theta\in\Theta.\end{array} \tag{3}$$

Unfortunately, as recalled hereinafter, if Θ contains a continuous subset of $\mathbb{R}$, then (3) leads to an integral equation (7) with no analytical solution in general. Therefore, since the work of Barankin [7], many studies quoted in [30]-[32] have been dedicated to the derivation of "computable" lower bounds approximating the MSE of the locally-best unbiased estimator (BB). All these approximations derive from sets of discrete or integral linear transform of the unbiasedness constraint (2b) and can be obtained using the following norm minimization lemma. Let $\mathbb{U}$ be an Euclidean vector space on $\mathbb{R}$ which has a scalar product $\langle\,|\,\rangle$. Let $(\mathbf{c}_1,\ldots,\mathbf{c}_K)$ be a free family of K vectors of $\mathbb{U}$ and $\mathbf{v}\in\mathbb{R}^K$. The problem of the minimization of $\|\mathbf{u}\|^2$ under the K linear constraints $\langle\mathbf{u}\mid\mathbf{c}_k\rangle = v_k$, $k\in[1,K]$ then has the solution:

$$\begin{array}{c}\min\left\{\|\mathbf{u}\|^2\right\} = \mathbf{v}^T\mathbf{G}^{-1}\mathbf{v} \quad\text{for}\quad \mathbf{u}_{opt} = \sum_{k=1}^{K}\alpha_k\mathbf{c}_k \\ \boldsymbol{\alpha} = \mathbf{G}^{-1}\mathbf{v},\quad \mathbf{G}_{n,k} = \langle\mathbf{c}_k\mid\mathbf{c}_n\rangle\end{array} \tag{4}$$

Indeed, let $\boldsymbol{\theta}^N = \left(\theta^1,\ldots,\theta^N\right)^T \in \Theta^N$ be a vector of N selected values of the parameter θ (aka test points), $\boldsymbol{\upsilon}_{\theta^0}\left(\mathbf{x};\boldsymbol{\theta}^N\right) = \left(\upsilon_{\theta^0}\left(\mathbf{x};\theta^1\right),\ldots,\upsilon_{\theta^0}\left(\mathbf{x};\theta^N\right)\right)^T$ be the vector of LR associated to $\boldsymbol{\theta}^N$, and $\xi(\theta) = \theta - \theta^0$ and $\boldsymbol{\xi}^N = \left(\xi\left(\theta^1\right),\ldots,\xi\left(\theta^N\right)\right)^T$. Then, any unbiased estimator $\widehat{\theta^0}(\mathbf{x})$ verifying (2b) must comply with

$$E_{\mathbf{x}|\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x}) - \theta^0\right)\boldsymbol{\upsilon}_{\theta^0}\left(\mathbf{x};\boldsymbol{\theta}^N\right)\right] = \boldsymbol{\xi}^N, \tag{5a}$$

and with any subsequent linear transformation of (5a). Thus, any given set of K ($K\leq N$) independent linear transformations of (5a):

$$E_{\mathbf{x}|\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x}) - \theta^0\right)\mathbf{h}_k^T\boldsymbol{\upsilon}_{\theta^0}\left(\mathbf{x};\boldsymbol{\theta}^N\right)\right] = \mathbf{h}_k^T\boldsymbol{\xi}^N, \tag{5b}$$

$\mathbf{h}_k\in\mathbb{R}^N$, $k\in[1,K]$, provides with a lower bound on the MSE (4):

$$MSE_{\theta^0}\left[\widehat{\theta^0}\right] \geq \left(\boldsymbol{\xi}^N\right)^T\mathbf{R}^\dagger_{\mathbf{H}_K}\boldsymbol{\xi}^N, \tag{5c}$$

where $\mathbf{R}^\dagger_{\mathbf{H}_K} = \mathbf{H}_K\left(\mathbf{H}_K^T\mathbf{R}_{\boldsymbol{\upsilon}_{\theta^0}}\mathbf{H}_K\right)^{-1}\mathbf{H}_K^T$, $\mathbf{H}_K = [\mathbf{h}_1\ \ldots\ \mathbf{h}_K]$ and $\left(\mathbf{R}_{\boldsymbol{\upsilon}_{\theta^0}}\right)_{n,m} = E_{\mathbf{x}|\theta^0}\left[\upsilon_{\theta^0}(\mathbf{x};\theta^m)\,\upsilon_{\theta^0}(\mathbf{x};\theta^n)\right]$. The BB is obtained by taking the supremum of (5c) over all the existing degrees of freedom $\left(N,\boldsymbol{\theta}^N,K,\mathbf{H}_K\right)$. All known bounds on the MSE deriving from the BB can be obtained with the appropriate instantiations of (5c). For example, the general class introduced lately in [31] is the limiting case of (5b-5c) [34] where $N\to\infty$ and $\boldsymbol{\theta}^N$ uniformly samples Θ, leading to :

$$E_{\mathbf{x}|\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x}) - \theta^0\right)\eta(\mathbf{x};\tau)\right] = \Gamma_h(\tau), \tag{6}$$
$$\eta(\mathbf{x};\tau) = \int_\Theta h(\tau,\theta)\,\upsilon_{\theta^0}(\mathbf{x};\theta)\,d\theta,\ \Gamma_h(\tau) = \int_\Theta h(\tau,\theta)\,\xi(\theta)\,d\theta,$$

where each $\mathbf{h}_k = \left(h\left(\tau_k,\theta^1\right),\ldots,h\left(\tau_k,\theta^N\right)\right)^T$ is the vector of samples of a parametric function $h(\tau,\theta)$, $\tau\in\Lambda\subset\mathbb{R}$. Then, the limiting case where $K\to\infty$ and the set $\{\tau_k\}_{k\in[1,K]}$ uniformly samples Λ yields the integral form of (5c) released in [31]:

$$\left|\begin{array}{l} MSE_{\theta^0}\left[\widehat{\theta^0}_{lmvu}(\mathbf{x})\right] = \int_\Lambda \Gamma_h(\tau)\,\beta(\tau)\,d\tau \\ \qquad\widehat{\theta^0}_{lmvu}(\mathbf{x}) - \theta^0 = \int_\Lambda \eta(\mathbf{x};\tau)\,\beta(\tau)\,d\tau \\ \int_\Lambda K_h(\tau',\tau)\,\beta(\tau)\,d\tau = \Gamma_h(\tau')\end{array}\right., \tag{7}$$

$$\begin{aligned}K_h(\tau,\tau') &= E_{\mathbf{x}|\theta^0}\left[\eta(\mathbf{x};\tau)\,\eta(\mathbf{x};\tau')\right] \\ &= \iint_{\Theta^2} h(\tau,\theta)\,R_{\upsilon_{\theta^0}}(\theta,\theta')\,h(\tau',\theta')\,d\theta d\theta',\end{aligned}$$
$$R_{\upsilon_{\theta^0}}(\theta,\theta') = E_{\mathbf{x}|\theta^0}\left[\frac{p(\mathbf{x}|\theta)}{p(\mathbf{x}|\theta^0)}\frac{p(\mathbf{x}|\theta')}{p(\mathbf{x}|\theta^0)}\right].$$

As mentioned above, in most practical cases, it is impossible to find an analytical solution of (7) to obtain an explicit form of the BB, which somewhat limits its interest. Nevertheless this formalism allows to use discrete (5b) or integral (6) linear transforms of the LR, possibly non-invertible, possibly optimized for a set of p.d.f. (such as the Fourier transform in [31]) to get a tight BB approximation.

3. LOWER BOUNDS FOR NON-STANDARD ESTIMATION

Non-standard deterministic estimation addresses the case where the conditional p.d.f. $p(\mathbf{x}|\theta)$ results from the marginalization of a conditional joint p.d.f. $p(\mathbf{x},\boldsymbol{\theta}_r|\theta)$ (1b) where $\Pi_{\boldsymbol{\theta}_r|\mathbf{x}}(\theta) = \left\{\boldsymbol{\theta}_r\in\mathbb{R}^{P_r} \mid p(\mathbf{x},\boldsymbol{\theta}_r|\theta) > 0\right\}$ is the support of $p(\boldsymbol{\theta}_r|\mathbf{x},\theta)$. The results introduced in the following are of interest if a closed-form of $p(\mathbf{x}|\theta)$ does not exist or if a closed-form of $p(\mathbf{x}|\theta)$ does exist but the resulting expression is intractable to derive lower bounds. If the supports of $p(\mathbf{x},\boldsymbol{\theta}_r|\theta)$ and $p(\boldsymbol{\theta}_r|\mathbf{x},\theta)$ are independent of θ: $\Delta(\theta) = \left\{(\mathbf{x},\boldsymbol{\theta}_r) \in \mathbb{C}^M \times \mathbb{R}^{P_r} \mid p(\mathbf{x},\boldsymbol{\theta}_r|\theta) > 0\right\} \triangleq \Delta$ and $\Pi_{\boldsymbol{\theta}_r|\mathbf{x}}(\theta) \triangleq \Pi_{\boldsymbol{\theta}_r|\mathbf{x}}$, then:

$$\begin{array}{c} p(\mathbf{x}|\theta) = \int\limits_{\Pi_{\boldsymbol{\theta}_r|\mathbf{x}}} p(\mathbf{x},\boldsymbol{\theta}_r|\theta)\, d\boldsymbol{\theta}_r, \\ E_{\mathbf{x},\boldsymbol{\theta}_r|\theta}\left[g(\mathbf{x},\boldsymbol{\theta}_r)\right] = E_{\mathbf{x},\boldsymbol{\theta}_r|\theta^0}\left[g(\mathbf{x},\boldsymbol{\theta}_r)\, \upsilon_{\theta^0}(\mathbf{x},\boldsymbol{\theta}_r;\theta)\right], \\ \upsilon_{\theta^0}(\mathbf{x},\boldsymbol{\theta}_r;\theta) = \frac{p(\mathbf{x},\boldsymbol{\theta}_r|\theta)}{p\left(\mathbf{x},\boldsymbol{\theta}_r|\theta^0\right)}, \end{array}$$

and (5a) can be reformulated as, $\forall n \in [1,N]$:

$$\begin{aligned} \theta^n - \theta^0 &= E_{\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x}) - \theta^0\right)\upsilon_{\theta^0}(\mathbf{x};\theta^n)\right] \\ &= \int\limits_{\Omega}\left(\widehat{\theta^0}(\mathbf{x}) - \theta^0\right) p(\mathbf{x}|\theta^n)\, d\mathbf{x} \\ &= \iint\limits_{\Delta}\left(\widehat{\theta^0}(\mathbf{x}) - \theta^0\right) p(\mathbf{x},\boldsymbol{\theta}_r|\theta^n)\, d\boldsymbol{\theta}_r d\mathbf{x} \\ \theta^n - \theta^0 &= E_{\mathbf{x},\boldsymbol{\theta}_r|\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x}) - \theta^0\right)\upsilon_{\theta^0}(\mathbf{x},\boldsymbol{\theta}_r;\theta^n)\right] \end{aligned}$$

that is, if $\boldsymbol{\upsilon}_{\theta^0}^T\left(\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}^N\right) = \left(\upsilon_{\theta^0}\left(\mathbf{x},\boldsymbol{\theta}_r;\theta^1\right),\ldots,\upsilon_{\theta^0}\left(\mathbf{x},\boldsymbol{\theta}_r;\theta^N\right)\right)$:

$$\begin{aligned} \boldsymbol{\xi}^N &= E_{\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x}) - \theta^0\right)\boldsymbol{\upsilon}_{\theta^0}\left(\mathbf{x};\boldsymbol{\theta}^N\right)\right] \\ &= E_{\mathbf{x},\boldsymbol{\theta}_r|\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x}) - \theta^0\right)\boldsymbol{\upsilon}_{\theta^0}\left(\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}^N\right)\right]. \end{aligned} \quad (8)$$

Since $E_{\mathbf{x}|\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x}) - \theta^0\right)^2\right] = E_{\mathbf{x},\boldsymbol{\theta}_r|\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x}) - \theta^0\right)^2\right]$,

$$\begin{array}{c} \min\left\{E_{\mathbf{x}|\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x}) - \theta^0\right)^2\right]\right\} \text{ under} \\ \boldsymbol{\xi}^N = E_{\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x}) - \theta^0\right)\boldsymbol{\upsilon}_{\theta^0}\left(\mathbf{x};\boldsymbol{\theta}^N\right)\right] \end{array} \quad (9a)$$

is equivalent to:

$$\begin{array}{c} \min\left\{E_{\mathbf{x},\boldsymbol{\theta}_r|\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x}) - \theta^0\right)^2\right]\right\} \text{ under} \\ \boldsymbol{\xi}^N = E_{\mathbf{x},\boldsymbol{\theta}_r|\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x}) - \theta^0\right)\boldsymbol{\upsilon}_{\theta^0}\left(\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}^N\right)\right] \end{array} \quad (9b)$$

Note that the equivalence between (9a) and (9b) holds only in $L^2(\Omega)$. Unfortunately the minimum norm lemma (4) provides the solution of (9b) in $L^2(\Delta)$, that is actually the solution of:

$$\begin{array}{c} \min\left\{E_{\mathbf{x},\boldsymbol{\theta}_r|\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x},\boldsymbol{\theta}_r) - \theta^0\right)^2\right]\right\} \text{ under} \\ \boldsymbol{\xi}^N = E_{\mathbf{x},\boldsymbol{\theta}_r|\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x},\boldsymbol{\theta}_r) - \theta^0\right)\boldsymbol{\upsilon}_{\theta^0}\left(\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}^N\right)\right] \end{array} \quad (9c)$$

leading to a looser lower bound, since $L^2(\Omega) \subset L^2(\Delta)$. The relationship between (9a-9c) shows that any approximation of the BB deriving from sets of discrete or integral linear transform of the unbiasedness constraint (5b)(6) has an analog formulation in non standard estimation obtained by substituting $E_{\mathbf{x},\boldsymbol{\theta}_r|\theta^0}[\,]$ for $E_{\mathbf{x}|\theta^0}[\,]$ and $\upsilon_{\theta^0}(\mathbf{x},\boldsymbol{\theta}_r;\theta)$ for $\upsilon_{\theta^0}(\mathbf{x};\theta)$. Actually, this is obtained simply by substituting $p(\mathbf{x},\boldsymbol{\theta}_r|\theta)$ for $p(\mathbf{x}|\theta)$ in any lower bound for standard estimation. In fact, we have simply performed an embedding of $L^2(\Omega)$ into $L^2(\Delta)$ for the search of the locally-best unbiased estimator. From this perspective, it seems appropriate to refer to these lower bounds as modified lower bounds (MLBs) as it has been proposed initially in [18][19] for the CRB in the restricted case where $p(\boldsymbol{\theta}_r|\theta) \triangleq p(\boldsymbol{\theta}_r)$. The result introduced here is general and holds whatever the p.d.f. of the random parameters depends or does not depend on the deterministic parameters. In the light of the above, the MCRB is obtained from the CRB:

$$E_{\mathbf{x}|\theta}\left[\left(\frac{\partial \ln p(\mathbf{x}|\theta)}{\partial\theta}\right)^2\right]^{-1} \to E_{\mathbf{x},\boldsymbol{\theta}_r|\theta}\left[\left(\frac{\partial \ln p(\mathbf{x},\boldsymbol{\theta}_r|\theta)}{\partial\theta}\right)^2\right]^{-1}$$

and for instance, the MBB is obtained from (6) and (7):

$$\begin{array}{c} MBB = \int\limits_{\Lambda}\Gamma_h(\tau)\,\beta(\tau)\, d\tau,\ \int\limits_{\Lambda} K_h(\tau',\tau)\,\beta(\tau)\, d\tau = \Gamma_h(\tau'), \\ \eta(\mathbf{x},\boldsymbol{\theta}_r;\tau) = \int\limits_{\Theta} h(\tau,\theta)\,\upsilon_{\theta^0}(\mathbf{x},\boldsymbol{\theta}_r;\theta)\, d\theta,\ \Gamma_h(\tau) = \int\limits_{\Theta} h(\tau,\theta)\,\xi(\theta)\, d\theta, \\ \begin{aligned} K_h(\tau,\tau') &= E_{\mathbf{x},\boldsymbol{\theta}_r|\theta^0}\left[\eta(\mathbf{x},\boldsymbol{\theta}_r;\tau)\,\eta(\mathbf{x},\boldsymbol{\theta}_r;\tau')\right] \\ &= \iint\limits_{\Theta^2} h(\tau,\theta)\, R_{\upsilon_{\theta^0}}(\theta,\theta')\, h(\tau',\theta')\, d\theta d\theta', \end{aligned} \\ R_{\upsilon_{\theta^0}}(\theta,\theta') = E_{\mathbf{x},\boldsymbol{\theta}_r|\theta^0}\left[\frac{p(\mathbf{x},\boldsymbol{\theta}_r|\theta)}{p\left(\mathbf{x},\boldsymbol{\theta}_r|\theta^0\right)}\frac{p\left(\mathbf{x},\boldsymbol{\theta}_r|\theta'\right)}{p\left(\mathbf{x},\boldsymbol{\theta}_r|\theta^0\right)}\right] \end{array}$$

3.1. A class of tighter modified lower bounds

Let $1_A(\boldsymbol{\theta}_r)$ denote the indicator function of subset A of $\mathbb{R}^{P_r}$. Then:

$$p(\mathbf{x}|\theta) = \int\limits_{\Pi_{\boldsymbol{\theta}_r|\mathbf{x}}} p(\mathbf{x},\boldsymbol{\theta}_r|\theta)\, d\boldsymbol{\theta}_r = \int\limits_{\mathbb{R}^{P_r}} p(\mathbf{x},\boldsymbol{\theta}_r|\theta)\, 1_{\Pi_{\boldsymbol{\theta}_r|\mathbf{x}}}(\boldsymbol{\theta}_r)\, d\boldsymbol{\theta}_r,$$

what can be rewritten as (after change of variables $\boldsymbol{\theta}_r = \boldsymbol{\theta}_r' + \mathbf{h}_r$):

$$p(\mathbf{x}|\theta) = \int\limits_{\mathbb{R}^{P_r}} p(\mathbf{x},\boldsymbol{\theta}_r + \mathbf{h}_r|\theta)\, 1_{\Pi_{\boldsymbol{\theta}_r|\mathbf{x}}}(\boldsymbol{\theta}_r + \mathbf{h}_r)\, d\boldsymbol{\theta}_r.$$

Therefore for any $\mathbf{h}_r$ such that:

$$1_{\Pi_{\boldsymbol{\theta}_r|\mathbf{x}}}(\boldsymbol{\theta}_r + \mathbf{h}_r) = 1_{\Pi_{\boldsymbol{\theta}_r|\mathbf{x}}}(\boldsymbol{\theta}_r),\ \forall \boldsymbol{\theta}_r \in \mathbb{R}^{P_r}, \quad (10)$$

then $p(\mathbf{x}|\theta) = \int\limits_{\Pi_{\boldsymbol{\theta}_r|\mathbf{x}}} p(\mathbf{x},\boldsymbol{\theta}_r + \mathbf{h}_r|\theta)\, d\boldsymbol{\theta}_r$ and, $\forall n \in [1,N]$:

$$\begin{aligned} \theta^n - \theta^0 &= \int\limits_{\Omega}\left(\widehat{\theta^0}(\mathbf{x}) - \theta^0\right) p(\mathbf{x}|\theta)\, d\mathbf{x} \\ &= \int\limits_{\Omega}\left(\widehat{\theta^0}(\mathbf{x}) - \theta^0\right) \int\limits_{\Pi_{\boldsymbol{\theta}_r|\mathbf{x}}} p(\mathbf{x},\boldsymbol{\theta}_r + \mathbf{h}_r|\theta^n)\, d\boldsymbol{\theta}_r d\mathbf{x} \end{aligned}$$

that is:

$$\begin{aligned} \boldsymbol{\xi}^N &= E_{\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x}) - \theta^0\right)\boldsymbol{\upsilon}_{\theta^0}\left(\mathbf{x};\boldsymbol{\theta}^N\right)\right] \\ \boldsymbol{\xi}^N &= E_{\mathbf{x},\boldsymbol{\theta}_r|\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x}) - \theta^0\right)\boldsymbol{\upsilon}_{\theta^0}\left(\mathbf{x},\boldsymbol{\theta}_r + \mathbf{h}_r;\boldsymbol{\theta}^N\right)\right]. \end{aligned}$$

The identity above means that in $L^2(\Omega)$ the two subsets of N constraints are equivalent system of linear equations yielding the same vector subspace of $L^2(\Omega)$: $span\left(\upsilon_{\theta^0}\left(\mathbf{x};\theta^1\right),\ldots,\upsilon_{\theta^0}\left(\mathbf{x};\theta^N\right)\right)$. Therefore in $L^2(\Omega)$ any set of $N \times K$ constraints of the form:

$$\boldsymbol{\xi}^N = E_{\mathbf{x},\boldsymbol{\theta}_r|\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x}) - \theta^0\right)\boldsymbol{\upsilon}_{\theta^0}\left(\mathbf{x},\boldsymbol{\theta}_r + \mathbf{h}_r^k;\boldsymbol{\theta}^N\right)\right], \quad (11)$$

where $\left\{\mathbf{h}_r^1,\ldots,\mathbf{h}_r^K\right\}$ satisfy (10), is equivalent to the set of N constraints (8). Fortunately this result does not hold a priori in $L^2(\Delta)$ where the $N \times K$ constraints (11) are expected to be independent (not necessarily true in the general case). The effect of adding constraints is to restrict the class of viable estimators $\widehat{\theta^0}(\mathbf{x},\boldsymbol{\theta}_r)$ and therefore to possibly increase the minimum norm obtained from (4):

$$\begin{array}{c} \min\left\{E_{\mathbf{x},\boldsymbol{\theta}_r|\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x},\boldsymbol{\theta}_r) - \theta^0\right)^2\right]\right\} \text{ under } k \leq K, \\ \boldsymbol{\xi}^N = E_{\mathbf{x},\boldsymbol{\theta}_r|\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x}) - \theta^0\right)\boldsymbol{\upsilon}_{\theta^0}\left(\mathbf{x},\boldsymbol{\theta}_r + \mathbf{h}_r^k;\boldsymbol{\theta}^N\right)\right] \end{array} \quad (12)$$

which remains smaller (or equal) than the minimum norm obtained on $L^2(\Omega)$ given by (9a). Note that the regularity condition (10) only imposes on $1_{\Pi_{\boldsymbol{\theta}_r|\mathbf{x}}}(\boldsymbol{\theta}_r)$, $\mathbf{x}\in\Omega$, to be of the following form:

$$1_{\Pi_{\boldsymbol{\theta}_r|\mathbf{x}}}(\boldsymbol{\theta}_r) = \left| \begin{array}{l} 0 \text{ if } \sum\limits_{\mathbf{h}_r\in F_{\mathbf{x}}}\sum\limits_{l\in\mathbb{Z}} 1_{\Pi^0_{\boldsymbol{\theta}_r|\mathbf{x}}}(\boldsymbol{\theta}_r + l\mathbf{h}_r) = 0, \\ 1, \text{otherwise}, \end{array}\right. \tag{13}$$

where $F_{\mathbf{x}}$ and $\Pi^0_{\boldsymbol{\theta}_r|\mathbf{x}}$ are subsets of $\mathbb{R}^{P_r}$. A typical example is the tighter MCRB obtained for the following set of constraints:

$$\begin{aligned} \mathbf{v} = d\theta\binom{0}{\mathbf{e}_1} &= E_{\mathbf{x},\boldsymbol{\theta}_r|\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x}) - \theta^0\right)\mathbf{c}(\mathbf{x},\boldsymbol{\theta}_r)\right], \\ \mathbf{c}(\mathbf{x},\boldsymbol{\theta}_r)^T &= \left(\upsilon_{\theta^0}\left(\mathbf{x},\boldsymbol{\theta}_r,\theta^0\right), \upsilon_{\theta^0}\left(\mathbf{x},\boldsymbol{\theta}_r,\theta^0+d\theta\right),\right. \\ &\quad \left.\upsilon_{\theta^0}\left(\mathbf{x},\boldsymbol{\theta}_r+\mathbf{u}_1 h_r^1,\theta^0\right),\ldots,\upsilon_{\theta^0}\left(\mathbf{x},\boldsymbol{\theta}_r+\mathbf{u}_{P_r}h_r^{P_r},\theta^0\right)\right) \end{aligned}$$

where $\mathbf{e}_1 = (1,0,\ldots,0)^T$ and $\mathbf{u}_k$ is the kth column of the identity matrix $\mathbf{I}_{P_r}$. If $d\theta, h_r^1,\ldots,h_r^{P_r}\to 0$, which imposes that (13) reduces to $\Pi_{\boldsymbol{\theta}_r|\mathbf{x}} = \mathbb{R}^{P_r}$, the lower bound obtained from (4) is:

$$\begin{aligned} \overline{MCRB}_{\theta^0} &= \mathbf{e}_1^T\mathbf{F}\left(\theta^0\right)^{-1}\mathbf{e}_1, \\ \mathbf{F}(\theta) &= E_{\mathbf{x},\boldsymbol{\theta}_r|\theta}\left[\frac{\partial\ln p(\mathbf{x},\boldsymbol{\theta}_r|\theta)}{\partial\left(\theta,\boldsymbol{\theta}_r^T\right)^T}\frac{\partial\ln p(\mathbf{x},\boldsymbol{\theta}_r|\theta)}{\partial\left(\theta,\boldsymbol{\theta}_r^T\right)}\right] \end{aligned} \tag{14}$$

Since $\mathbf{F}(\theta) = \begin{bmatrix} f_\theta(\theta) & \mathbf{f}^T_{\boldsymbol{\theta}_r,\theta}(\theta) \\ \mathbf{f}_{\boldsymbol{\theta}_r,\theta}(\theta) & \mathbf{F}_{\boldsymbol{\theta}_r}(\theta)\end{bmatrix}$, therefore:

$$\begin{aligned} \overline{MCRB}_{\theta^0} &= \frac{1}{f_\theta\left(\theta^0\right) - \mathbf{f}^T_{\boldsymbol{\theta}_r,\theta}\left(\theta^0\right)\mathbf{F}^{-1}_{\boldsymbol{\theta}_r}\left(\theta^0\right)\mathbf{f}_{\theta,\boldsymbol{\theta}_r}\left(\theta^0\right)} \geq \\ &\qquad \frac{1}{f_\theta\left(\theta^0\right)} = MCRB_{\theta^0} \end{aligned} \tag{15}$$

3.2. Another class of tighter modified lower bounds

Any real-valued function $\psi(\mathbf{x},\boldsymbol{\theta}_r;\theta)$ which support is Δ satisfying:

$$\int_{\Pi_{\boldsymbol{\theta}_r|\mathbf{x}}}\psi(\mathbf{x},\boldsymbol{\theta}_r;\theta)\,d\boldsymbol{\theta}_r = 0, \tag{16}$$

is a Bayesian lower bound-generating function [35], such as:

$$\psi_s^{\mathbf{h}_r}(\mathbf{x},\boldsymbol{\theta}_r;\theta) = \left(\frac{p(\boldsymbol{\theta}_r+\mathbf{h}_r|\mathbf{x},\theta)}{p(\boldsymbol{\theta}_r|\mathbf{x},\theta)}\right)^m - \left(\frac{p(\boldsymbol{\theta}_r-\mathbf{h}_r|\mathbf{x},\theta)}{p(\boldsymbol{\theta}_r|\mathbf{x},\theta)}\right)^{1-m} \tag{17}$$

where $m\in]0,1[$, yielding the Weiss-Weinstein bound. Let $\boldsymbol{\psi}(\mathbf{x},\boldsymbol{\theta}_r;\theta)$ be a vector of L independent functions satisfying (16). Then:

$$E_{\mathbf{x}|\theta^0}\left[\begin{pmatrix}\widehat{\theta^0}(\mathbf{x}) - \theta^0 \\ \boldsymbol{\upsilon}_{\theta^0}\left(\mathbf{x};\boldsymbol{\theta}^N\right)\end{pmatrix}\boldsymbol{\psi}\left(\mathbf{x},\boldsymbol{\theta}_r;\theta^0\right)^T\right] = \mathbf{0}, \tag{18}$$

which means that [33, Lemma 2] the addition of the set of L constraints $E_{\mathbf{x}|\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x}) - \theta^0\right)\boldsymbol{\psi}\left(\mathbf{x},\boldsymbol{\theta}_r;\theta^0\right)\right] = \mathbf{0}$ to any linear transformation of (5a) does not change the associated lower bound (5c). Fortunately, once again, this result does not hold a priori in $L^2(\Delta)$ for most choices of $\boldsymbol{\psi}(\mathbf{x},\boldsymbol{\theta}_r;\theta)$. Therefore one can possibly increases the minimum norm obtained from (9c) by computing:

$$\begin{aligned} &\min\left\{E_{\mathbf{x},\boldsymbol{\theta}_r|\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x},\boldsymbol{\theta}_r) - \theta^0\right)^2\right]\right\} \text{ under} \\ &\boldsymbol{\xi}^N = E_{\mathbf{x},\boldsymbol{\theta}_r|\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x},\boldsymbol{\theta}_r) - \theta^0\right)\boldsymbol{\upsilon}_{\theta^0}\left(\mathbf{x},\boldsymbol{\theta}_r;\boldsymbol{\theta}^N\right)\right] \\ &\quad \mathbf{0} = E_{\mathbf{x},\boldsymbol{\theta}_r|\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x},\boldsymbol{\theta}_r) - \theta^0\right)\boldsymbol{\psi}\left(\mathbf{x},\boldsymbol{\theta}_r;\theta^0\right)\right] \end{aligned} \tag{19}$$

which remains smaller (or equal) than the minimum norm obtained on $L^2(\Omega)$ given by (9a). First note that it is generally not possible to compare (12) with (19) since they derive from different subset of constraints. Second, (19) can be used with joint p.d.f. $p(\mathbf{x},\boldsymbol{\theta}_r|\theta)$ which does not satisfy the regularity condition (13) since functions (17) are essentially free of regularity conditions [35]. Another tighter MCRB is obtained as the limiting case ($d\theta\to 0$) resulting from:

$$\begin{aligned} \mathbf{v} = d\theta\binom{0}{\mathbf{e}_1} &= E_{\mathbf{x},\boldsymbol{\theta}_r|\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x}) - \theta^0\right)\mathbf{c}(\mathbf{x},\boldsymbol{\theta}_r)\right], \\ \mathbf{c}(\mathbf{x},\boldsymbol{\theta}_r)^T &= \left(\upsilon_{\theta^0}\left(\mathbf{x},\boldsymbol{\theta}_r,\theta^0+d\theta\right), \upsilon_{\theta^0}\left(\mathbf{x},\boldsymbol{\theta}_r,\theta^0\right), \boldsymbol{\psi}\left(\mathbf{x},\boldsymbol{\theta}_r;\theta^0\right)^T\right), \\ \overline{MCRB}_{\theta^0} &= \mathbf{e}_1^T E_{\mathbf{x},\boldsymbol{\theta}_r|\theta^0}\left[\begin{pmatrix}\frac{\partial\ln p(\mathbf{x},\boldsymbol{\theta}_r|\theta)}{\partial\theta} \\ \boldsymbol{\psi}(\mathbf{x},\boldsymbol{\theta}_r;\theta)\end{pmatrix}\begin{pmatrix}\frac{\partial\ln p(\mathbf{x},\boldsymbol{\theta}_r|\theta)}{\partial\theta} \\ \boldsymbol{\psi}(\mathbf{x},\boldsymbol{\theta}_r;\theta)\end{pmatrix}^T\right]^{-1}\mathbf{e}_1 \end{aligned}$$

3.3. The relationship with hybrid lower bounds

The tightest modified lower bounds are obtained by combination of constraints (12)(19) as the solution of:

$$\begin{aligned} &\min\left\{E_{\mathbf{x},\boldsymbol{\theta}_r|\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x},\boldsymbol{\theta}_r) - \theta^0\right)^2\right]\right\} \text{ under } k\leq K, \\ &\boldsymbol{\xi}^N = E_{\mathbf{x},\boldsymbol{\theta}_r|\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x}) - \theta^0\right)\boldsymbol{\upsilon}_{\theta^0}\left(\mathbf{x},\boldsymbol{\theta}_r+\mathbf{h}_r^k;\boldsymbol{\theta}^N\right)\right], \\ &\quad \mathbf{0} = E_{\mathbf{x},\boldsymbol{\theta}_r|\theta^0}\left[\left(\widehat{\theta^0}(\mathbf{x},\boldsymbol{\theta}_r) - \theta^0\right)\boldsymbol{\psi}\left(\mathbf{x},\boldsymbol{\theta}_r;\theta^0\right)\right], \end{aligned} \tag{20}$$

where $\boldsymbol{\psi}\left(\mathbf{x},\boldsymbol{\theta}_r;\theta^0\right)$ satisfies (16). First, they can be applied only to problems where the support $\Pi_{\boldsymbol{\theta}_r|\mathbf{x}}$ satisfies (13). Therefore, if the existence of a MCRB of the form (14) is required, then necessarily $\Pi_{\boldsymbol{\theta}_r|\mathbf{x}} = \mathbb{R}^{P_r}$. Second, the modified lower bound obtained is lower than or equal to (9a). As an example, let us consider the situation where $\Pi_{\boldsymbol{\theta}_r|\mathbf{x}} = \mathbb{R}^{P_r}$ and let $\Lambda_\theta = \{h\in\mathbb{R} \mid \theta+h\in\Theta\}$ and $\Lambda_{\boldsymbol{\theta}_r} = \left\{\mathbf{h}_r\in\mathbb{R}^{P_r} \mid \boldsymbol{\theta}_r+\mathbf{h}_r\in\Pi_{\boldsymbol{\theta}_r}\right\}$. Then if we choose $\psi(\mathbf{x},\boldsymbol{\theta}_r;\theta)_l = \psi_{m_l}^{\mathbf{h}_r^l}(\mathbf{x},\boldsymbol{\theta}_r;\theta)$ (17), $1\leq l\leq L$, and a set of test points of the form $\left(\theta^0,\theta^0+h^1,\ldots,\theta^0+h^N\right)$, then the tightest modified lower bound solution of (20) is given by:

$$\mathbf{MLB}\left(\theta^0\right) = \sup_{\{h^n\}\in\Lambda_{\theta^0},\left\{\mathbf{h}_r^l\right\}\in\Lambda_{\boldsymbol{\theta}_r},\{m_l\}\in]0,1[}\left\{\mathbf{v}^T\mathbf{G}^{-1}\mathbf{v}\right\} \tag{21}$$

where $\mathbf{v} = \left(h^1,\ldots,h^N,0,\ldots,0\right)^T$ and $\mathbf{G}$ is given by $\mathbf{V}$ (15-19) in [36]. Obviously, the $\mathbf{MLB}\left(\theta^0\right)$ (21) is the special case of the HMSSWB [36] where the vector of hybrid parameters reduce to the deterministic parameters (no random parameters). As shown in [36], the HMSSWB encompasses the hybrid lower bounds based on linear transformation on the centered likelihood-ratio (CLR) function [37] which is the cornerstone to generate a large class of hybrid bounds including any existing approximation of the MBB.

3.4. On the tightness of modified lower bounds

The "tightness condition" required to obtain a modified lower bound as tight as the standard lower bound is simply: it is necessary, and sufficient, that the estimator solution of the norm minimization under linear constraints (4)(12)(19)(20) belongs to $L^2(\Omega)$. For example, if we consider the $\overline{MCRB}_{\theta^0}$ (14) then the tightness condition is (4):

$$\begin{aligned} &\frac{\partial\ln p(\mathbf{x},\boldsymbol{\theta}_r|\theta)}{\partial\theta} - \mathbf{f}_{\boldsymbol{\theta}_r,\theta}(\theta)\mathbf{F}^{-1}_{\boldsymbol{\theta}_r}(\theta)\frac{\partial\ln p(\mathbf{x},\boldsymbol{\theta}_r|\theta)}{\partial\boldsymbol{\theta}_r} \in L^2(\Omega) \\ &\Leftrightarrow\quad \frac{\partial\ln p(\mathbf{x},\boldsymbol{\theta}_r|\theta)}{\partial\theta\partial\boldsymbol{\theta}_r^T} - \mathbf{f}_{\boldsymbol{\theta}_r,\theta}(\theta)\mathbf{F}^{-1}_{\boldsymbol{\theta}_r}(\theta)\frac{\partial\ln p(\mathbf{x},\boldsymbol{\theta}_r|\theta)}{\partial\boldsymbol{\theta}_r\partial\boldsymbol{\theta}_r^T} = \mathbf{0} \end{aligned}$$

which has been introduced in [38, (34)] at the expense of a quite complex proof.

4. REFERENCES

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